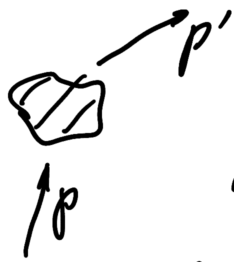
Impurity scattering.

Consider electrons moving in a random potential, created by randomly located impurities

$$U(\vec r)=\sum_n u(\vec r-\vec r_n)$$



Now a quasiparticle with momentum $\vec p$ scatters into a state with momentum $\vec p\,'$

Fermi golden rule: $\Gamma_{\vec p\to\vec p\,'}=\frac{2\pi}{\hbar}\sum_{\vec p\,'}|U_{\vec p\vec p\,'}|^2\delta(\xi_{\vec p}-\xi_{\vec p\,'})$

To find the average matrix element $|U_{\vec p\vec p\,'}|^2$, we have to use that $|\vec p\rangle=\frac{1}{\sqrt V}e^{i\vec p\vec r/\hbar}$

$$\langle\vec p|U|\vec p\,'\rangle=\frac{1}{V}\sum_i\int e^{-i\vec p\vec r}u(\vec r-\vec r_i)e^{i\vec p\,'\vec r\,'/\hbar}d\vec r\,'=$$

$$=\frac{1}{V}\sum_i e^{-i\vec r_i(\vec p-\vec p\,')}u_{\vec p\vec p\,'}$$

Note: often $\hbar=1$

$$|\langle\vec p|U|\vec p\,'\rangle|^2=\frac{1}{V^2}\sum_{i,j}e^{i(\vec r_i-\vec r_j)(\vec p-\vec p\,')/\hbar}|u_{\vec p\vec p\,'}|^2$$

The scattering rate

$$\Gamma_{\vec p\to\vec p\,'}=\frac{2\pi}{\hbar}|u_{\vec p\vec p\,'}|^2\sum_{i,j}e^{\frac{i}{\hbar}(\vec p-\vec p\,')(\vec r_i-\vec r_j)}\cdot\frac{1}{V^2}$$

Collision integral

$$St\, f=\sum_{\vec p\,'}\Gamma_{\vec p\to\vec p\,'}\overbrace{[-f_{\vec p}(1-f_{\vec p\,'})+f_{\vec p\,'}(1-f_{\vec p})]}^{=f_{\vec p\,'}-f_{\vec p}}$$

(For simplicity the distributions may be considered

(For simplicity the distributions may be considered uniform)

$$St\, f = \frac{1}{V^2}\frac{2\pi}{\hbar}\sum_{\vec{p}'}\sum_{i,j} e^{i\frac{(\vec{p}-\vec{p}')(\vec{r}_i-\vec{r}_j)}{\hbar}} (f_{\vec{p}}-f_{\vec{p}'})|U_{\vec{p}\vec{p}'}|^2\,\delta(\xi_{\vec{p}}-\xi_{\vec{p}'})$$

Summation wrt $\vec{p}'$ may be replaced by integration: $\frac{1}{V}\sum_{\vec{p}'} \dots = \int \frac{d\vec{p}'}{(2\pi\hbar)^3} \dots$

Only $i=j$ contribute, because the other contributions oscillate fast when integrated over $\vec{p}'$

Averaging over the locations of impurities:

$$\langle \dots \rangle_i = \frac{1}{V}\int d\vec{r}_i \dots$$

Now see how the distribution function relaxes

$$St\, f = \frac{2\pi}{\hbar} n \int \frac{d^3\vec{p}'}{(2\pi)^3}\left[-f_{\vec{p}}(1-f_{\vec{p}'}) + f_{\vec{p}'}(1-f_{\vec{p}})\right]\delta(\xi_{\vec{p}}-\xi_{\vec{p}'})\,|U_{\vec{p}\vec{p}'}|^2$$

$$= \frac{2\pi}{\hbar} n \int \frac{d^3\vec{p}'}{(2\pi)^3}\left[f_{\vec{p}'}-f_{\vec{p}}\right]\delta(\xi_{\vec{p}}-\xi_{\vec{p}'})\,|U_{\vec{p}\vec{p}'}|^2$$

$$= \frac{2\pi}{\hbar} n \int \frac{dS}{(2\pi)^3}\frac{1}{v_{\vec{p}'}}|U_{\vec{p}\vec{p}'}|^2 (f_{\vec{p}'}-f_{\vec{p}})$$

Integration over the Fermi surface

Now solve the kinetic equation. Assume that an external electric field E slightly modifies the distribution function:

$$f = f_0 + f_1$$

The scattering amplitude depends only on the

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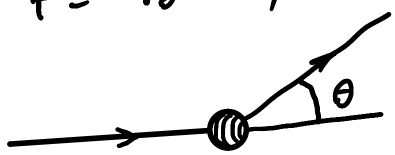


The scattering amplitude depends only on the angle between the initial direction of motion and that of the scattered state

Introduce a new rotation [That part may be skipped]

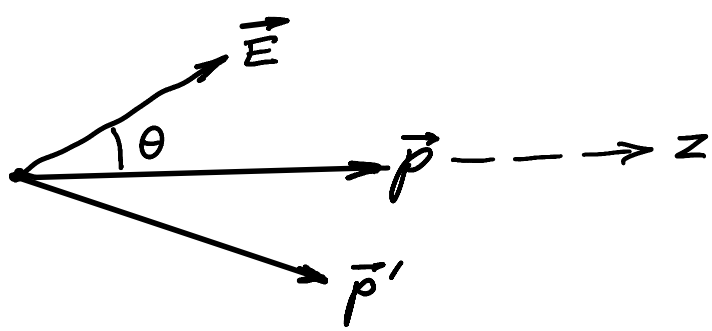
$$\nu_1(\varepsilon) = \int \frac{d^3\vec{p}\,'}{(2\pi)^3}\,\delta(\xi_p - \xi_{p'}) = \frac{p^2}{2\pi^2 v}$$ – DoS per spin

If we take into account spins, multiply this by 2

Then $$\mathrm{St}\, f = \int W(\theta)\,[f_1(\vec{p}\,') - f_1(\vec{p})]\,\frac{d\Omega}{4\pi}$$

Ansatz for f_1: $$f_1(\vec{p}) = \eta(\varepsilon)\,\vec{E}\vec{p}$$

$$\mathrm{St}\, f = pE\,\eta(\varepsilon)\int W(\theta)\left[\cos(\widehat{\vec{p}\,',\vec{E}}) - \cos(\widehat{\vec{p},\vec{E}})\right]\frac{d\Omega}{4\pi}$$



$$\vec{p}\,'\vec{E} = \underbrace{p'_z}_{=\,p\cos(\vec{p}\,',\vec{p})}\,\underbrace{E_z}_{=\,E\cos(\vec{p},\vec{E})} + \underbrace{p'_\perp E_\perp}$$ ← This contribution vanishes upon integration

Then
$$\mathrm{St}\, f = \vec{p}\vec{E}\,\eta(\varepsilon)\int W(\theta)\,[\cos\theta - 1]\,\frac{d\Omega}{2\pi}$$

Thus, $$\mathrm{St}\, f = -\frac{f_1}{\tau}$$

Where $$\tau^{-1} = \frac{2\pi}{\hbar}\, n \int |u_{\vec{p}\vec{p}'}|^2\,\delta(\varepsilon_{\vec{p}} - \varepsilon_{\vec{p}'})\,(1-\cos\theta)\,\frac{d\vec{p}\,'}{(2\pi\hbar)^3}$$

Where $\tau^{-1} = \frac{2\pi}{\hbar} n \int |u_{\vec{p}\vec{p}'}|^2 \delta(\varepsilon_{\vec{p}} - \varepsilon_{\vec{p}'})(1-\cos\theta) \frac{d\vec{p}'}{(2\pi\hbar)^3}$

$$= \frac{2\pi}{\hbar} n \int |u(\theta)|^2 \, \nu(\varepsilon) \, (1-\cos\theta) \frac{d\Omega}{4\pi}$$

This is the <u>transport</u> scattering time

<u>Conductivity</u> derivation

$$e\vec{E} \frac{\partial f_0}{\partial \vec{p}} = -\frac{f-f_0}{\tau}$$

Note: we replaced f by f_0 in the $l h s$

$$\frac{\partial f_0}{\partial \vec{p}} = \frac{\partial \varepsilon}{\partial \vec{p}} \frac{\partial f_0}{\partial \varepsilon} = \vec{v} \frac{\partial f_0}{\partial \varepsilon}$$

Then $f - f_0 = -\tau e \vec{E} \vec{v} \frac{\partial f_0}{\partial \varepsilon}$

$$\vec{j} = 2e \int \vec{v} f \frac{d^3p}{(2\pi\hbar)^3} = 2e \int \vec{v} (f - f_0) \frac{d^3p}{(2\pi\hbar)^3}$$

Spin

$$= -e^2 \int \vec{v} (\vec{v}\vec{E}) \tau \underbrace{\frac{\partial f_0}{\partial \varepsilon}}_{\approx -\delta(\varepsilon - \varepsilon_F)} \nu(\varepsilon) \, d\varepsilon \frac{d\Omega}{4\pi} =$$

$$= \frac{1}{3} e^2 v^2 \tau \nu(\varepsilon_F) \cdot E$$

$$\rightarrow \boxed{\sigma = \frac{1}{3} e^2 v^2 \tau \, \nu(\varepsilon_F)}$$

$\nu(\varepsilon_F) = \frac{p_F^2}{\pi^2 v_F}$ (including spin)

$$\sigma = \frac{e^2 v \tau p_F^2}{3\pi^2}$$

Note that $\sigma \sim p_F (p_F l)$ in 3D

$\sigma \sim p_F l$ in 2D

Assume, the impurities are short-range scatterers:

$u_i(\vec{r}) = u_0 \, \delta(\vec{r} - \vec{r}_i)$

Then $\tau^{-1} = \frac{2\pi}{\hbar} n u_0^2$

In general, one should not confuse the transport scattering time

$$\frac{1}{\tau_{tr}} = \frac{2\pi}{\hbar} n \int |u_{\vec{p}\vec{p}'}|^2 \, \delta(\varepsilon_{\vec{p}} - \varepsilon_{\vec{p}'}) \, (1 - \cos\theta) \, \frac{d^3\vec{p}'}{(2\pi\hbar)^3}$$

and the elastic scattering

$$\frac{1}{\tau_0} = \frac{2\pi}{\hbar} n \int |u_{\vec{p}\vec{p}'}|^2 \, \delta(\varepsilon_{\vec{p}} - \varepsilon_{\vec{p}'}) \frac{d^3\vec{p}'}{(2\pi\hbar)^3}$$